4 RELATIONSHIPS THAT WILL INCREASE YOUR STUDENTS' NUMBER SENSE

CHRISTINA TONDEVOLD

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That Will Increase Your Students' Number Sense

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Cover design by CreativeIdeasDesigns

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Introduction

The Learning of Addition Facts

A lot of emphasis has been placed on children being fast with their math facts. The National Mathematics Advisory Panel (2008) recommends that children be proficient with addition and subtraction by the end of 3rd grade. However, Common Core State Standards (National Governors Association, 2010) lays out this fluency track for addition and subtraction facts:

Grade level Standard	Fluency expectation by the end of the grade
K.OA.5	Fluently add and subtract within 5.
1.OA.6	Add and subtract within 20 using strategies. Fluently add and subtract within 10.
2.OA.2	Use strategies fluently to add and subtract within 20. Know from memory sums of two one-digit addends.

People do not disagree whether or not children need to be fluent with their facts. If a child is not fluent, it makes mathematics cumbersome as they progress through to more difficult mathematics. However, where people disagree is what is 'fluent' and what is the best path to help children reach a level of fluency. Susan Jo Russell (2000) states that fluency has three components:

- **Efficiency -** children need to have a quick way to obtain the answer.
- **Accuracy -** they need to get the right answer.
- **Flexibility -** if they can not figure it out, they have another way to approach the problem.

Learning your 'facts' can be done by memorization. But if we want children to be FLUENT with their facts, then we need to focus on developing their flexibility, not just their memorization capabilities. We can memorize anything for a short period of time but what we should really be trying to get our children to do is RECALL their facts. When a child cannot instantly pull a 'fact' out of his memory, he will revert back to counting on his fingers if he does not have a flexibility with numbers. So, how do we recall information quickly if it is not 'memorized'? The more connections we make to information the easier it is to recall that piece of information. If all a child knows about 6 + 7 is that 6+7=13, they do not have a fully developed understanding of 6 + 7; their understanding is unilateral.

Here are some of the things children need to know about any ‘basic math fact’ (let’s use 6 + 7 as an example) as part of developing a full understanding that will aid in their *recall* of the facts, not just memorization (a larger set of understandings is shown in Figure 1.1):

- They can relate the abstract symbols to a real problem, a context. Like when a family of 6 invites a family of 7 over for dinner and they have to figure out how many chairs they are going to need.
- They can model the problem and count to determine the answer. Children start by *counting all* (also known as Direct Modeling) where they need to count out six chairs, count out 7 chairs and then go back and count them all. Then children naturally start to see that they could just count on; they keep the 6 in their mind and count on 7 more. (Carpenter, et. al, 2014)
- Numbers relate to each other and they can use those relationships to help them add. Carpenter (2014) calls this using Derived Facts; using a fact you know to help you with one you do not know. A child who comes from a football loving family may see 6 + 7 and think *“That’s like two touchdowns (7 points and 7 points) which is 14, but it’s just one less.”* Or in the context of the family get-together, they might have a table that fits 10 people and they see that all six of their family members can fit along with 4 from the other family. But that leaves three people out, so they can see how 6+7 is equal to 10+3.

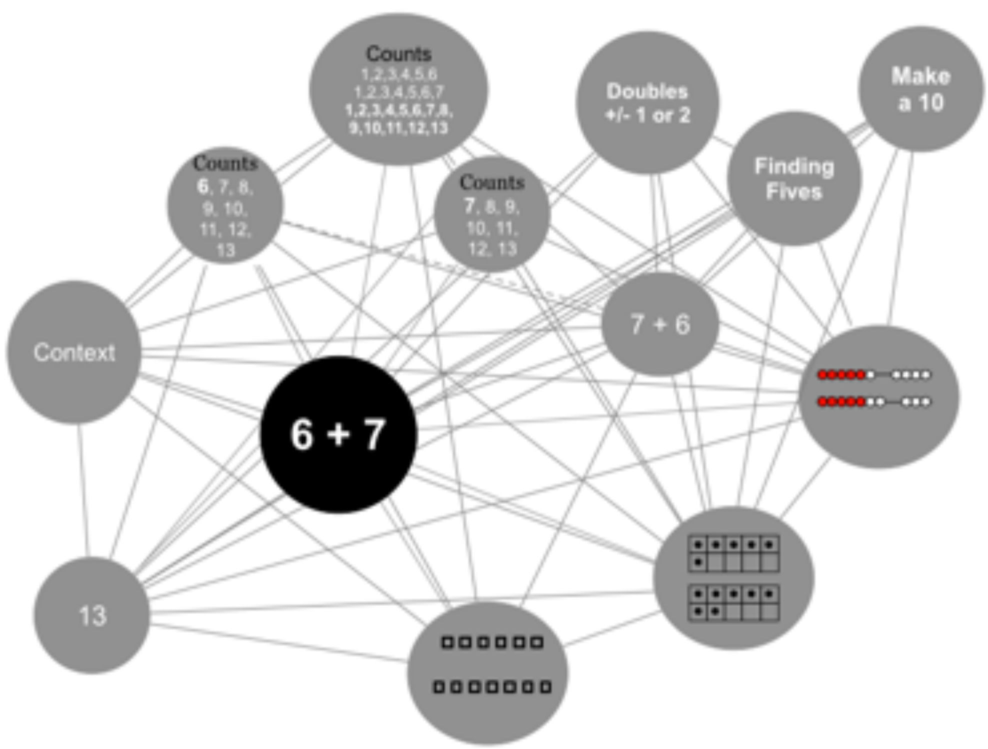


Figure 1.1 Set of connections that can be made around the fact 6 + 7.

Baroody (2006) states, *"recent research supports the view that the basic number-combination knowledge of mental-arithmetic experts is not merely a collection of isolated or discrete facts but rather a web of richly interconnected ideas."*(p. 26) When children develop a connected web, they will have a rich understanding around a 'fact' which helps them be more flexible and be able to recall the information when needed. That web of understanding also aids in the transfer of information to other concepts in mathematics. As children gain new knowledge, this web grows larger and larger. For instance, let's take the understanding that 6 + 7 is equal to 10 + 3. This comes from two big ideas around addition; compensation and part/whole relationships (Fosnot & Dolk, 2001).

Understanding those ideas allows children to see how when two parts come together to create a whole, the parts can be changed yet the whole can stay the same. 6 + 7 = 8 + 5 = 10 + 3 = 13, because each time we take an amount from one of the addends we *compensate* for that by adding that same amount to the other addend. This does not seem like that big of a deal when working with 'basic facts,' and many people say all the time, *"Why do kids need to learn all this stuff about their facts?! Just have them memorize!"* We could. However, when we do that, children see each fact as an isolated fact without any connections to other facts, connections to problems in context, or connections to addition problems with larger numbers.

I was the kid who was good at memorization. I was good at following procedures, and, therefore, I thought I was good at math. But, I really was not good at mathematics. Give me a story problem, I struggled. Give me a multi-digit addition problem, I had to do it with paper and pencil. I was horrible at doing mental math because all I could do was the algorithm, and that is hard to keep track of in your head. I had no flexibility with numbers and so when I was given a problem like 56 + 37, I did not relate it to 6 + 7, the only way I could do the problem was to line them up and do the algorithm. While my husband was relating it to 6 + 7, he quickly manipulated the numbers in his head and got the answer before I could write down the problem! He was using the derived facts he uses on 'basic facts' to help him with the multi-digit addition. He thinks about it as getting to a '10' just like he would if the problem was 6 + 7, "56, just take 4 from the 37 and give to 56. That makes 60 and 33…so 93." What he had that I did not have was number sense. Number sense starts with the numbers 0 through 10, and the understandings that are developed around those numbers help children become *flexibly* fluent with larger numbers as well.

Why This Book Focuses on Building Number Sense

Howden (1989) describes number sense as a *"…good intuition about number and their relationships. It develops gradually as a result of exploring numbers, visualizing them in a variety of contexts, and relating them in ways that are not limited by traditional algorithms."* (p.11) That quote is the most often cited quote about number sense. I love the portion of the quote that talks about how it is developed *"…gradually as a result of exploring numbers, visualizing them in a variety of contexts, and relating them in ways that are not limited by traditional algorithms."* That part has always made sense to me. We can not stand in the front of the classroom and tell children, "Four is one less than five," and expect them to understand it. It develops over time, as children play games; one of them rolls a 5 and one rolls a 4 and they get to see that the person who rolled the 5 moves one more space. It comes from playing the old game of War;

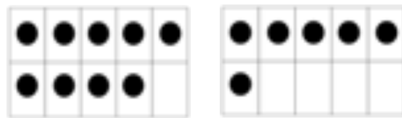
one player flips over a 5, the other flips over a 4, the children realize that 5 is more than 4 as that player takes both of the cards. It comes from just counting, realizing that 5 is always right after 4. Number sense cannot be taught, it is caught. Children 'catch' number sense through an exploration with numbers. We can not tell children the relationships and expect them to learn them, they learn by doing. They learn by playing. They learn by us providing experiences that allow them to explore and see the relationships around numbers. For some children, those relationships are not obvious and so each activity includes 'Look For' information. Those boxes are there to help remind you to be watching for particular understandings as children start to develop the number relationships.

Howden makes reference to the number relationships in the quote *"a good intuition about number* ***and their relationships****."* So what exactly does that look like? What are those relationships? I wanted it laid out for me. What is it that children who have number sense understand about numbers compared to children who do not have number sense? After years of trying to figure it out, I found the answer in John Van de Walle's book Teaching Student Centered Mathematics K-3 (2006). I originally read the book in 2007 (there is a new version out in 2014) and when I came across these four number relationships he mentions in the book a lightbulb came on for me:

1. **Spatial relationships –** recognizing how many without counting by seeing a visual pattern.
2. **One and two more, one and two less –** this is not the ability to count on two or count back two, but instead knowing which numbers are one and two less or more than any given number.
3. **Benchmarks of 5 and 10 –** ten plays such an important role in our number system (and two fives make up 10), children must know how numbers relate to 5 and 10.
4. **Part-Part-Whole –** seeing a number as being made up of two or more parts.

At first I did not understand the full value of these four relationships and especially why we needed to develop them with small numbers. It was not until I started working with PreK-2 children around these four relationships that I realized this was the number sense that some kids were lacking. These relationships may seem trivial, even unimportant. However, if children do not have these four understandings around small numbers it affects their flexibility, and without flexibility they will not become truly fluent. Think about the problem 9 + 6, if all a child knows about that is 9 + 6 = 15 but they cannot quickly pull that fact from memory then their only alternative is to count on their fingers. However, if they have developed the four number relationships above, they can quickly come up with an alternative strategy. A common strategy on that problem is thinking of it as 10 + 6 then subtract 1. For children who do not have number sense they can not begin to follow the logic of the child who came up with this strategy. So, let's take a brief look at how those four relationships play a roll in helping children develop the understanding that 9 + 6 = (10 + 6) - 1 (each section of this book focuses on one of these relationships, so they will be described in more detail there.

Spatial Relationship
The section about developing spatial relationships helps give children a visual to go along with a number instead of just the abstract numeral. Common visuals are dot patterns and ten frames, so a child might have this visual using ten frames in their mind for 9 + 6:



One/Two More and Less
The spatial relationship, as seen in the ten frames above, help build the idea of One/Two More and Less in this problem because it gives children a concrete way to see that 9 is just one less than 10 because the ten frame is missing one.

Benchmarks of 5 and 10
If the objects are placed into the ten frame sequentially (as they are in the picture above), it helps children develop an understanding of how that number relates to 5. No matter how objects are placed in a ten frame, the visual created helps children see how that amount relates to the benchmark of 10. This aids in helping children understand and use strategies based on using 10, because they are able to see that if the ten frame was full it would create the problem 10 + 6, but they need to take one away because it only had 9 instead of the full ten.

Part-Part-Whole
For too many children, 6 is just 6. They can not see how it can be broken down into parts that create the whole of six. Without that understanding, children will not decompose (break apart) numbers to help them make friendlier numbers. Some children may like creating a 10 when one of the addends is 9 but they do not like having to subtract one at the end. So, children often think about 9 + 6 as (9 + 1) + 5. The ability to use that strategy efficiently is dependent on a lot of these four relationships but without the understanding that the 6 can be decomposed into a 5 and a 1, the strategy would never even come to mind.

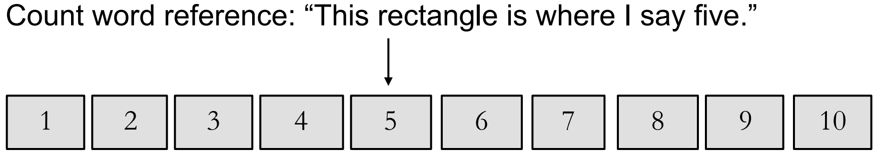
So why are these relationships so important to develop before we expect children to be fluent? If we jump straight to having children memorize their math facts (which is what some people see as being fluent) we end up with high school children still counting on their fingers. This book was not what I first started out to write. I had planned to write a book to help teachers develop Derived Facts with their students. However, as I started that book I realized that most of the activities in the book required children to have number sense first. So, I could not expect students to jump into thinking about numbers in these different ways until they developed number sense. As Ann Dowker (1992) states,*"To the person without number sense, arithmetic is a bewildering territory in which any deviation from the known path may rapidly lead to being totally lost."* (p.52) It is important that children develop a strong understanding of numbers and their relationships before we ask them to compute. If they do not have that strong foundation, the only approaches they will have for addition are counting on their fingers or rote memorization. Neither of those are acceptable. We want children to have a foundation of number sense which allows them to recall and transfer information with ease. That does not happen if we move to memorization and drill too quickly. Brownell and Chazal (1935) found that if we move to an emphasis on speed too soon it just encourages children to become faster at their informal approaches (i.e. counting on their fingers). During the drill, they are just practicing their inefficient and sometimes inaccurate strategies, which leads us to think we need to drill them more. However, Rathmell (1978) showed that developing children's derived fact strategies lead to better retention of their basic facts. So the goal is to facilitate children's development of derived facts, but we can not expect them to develop those if they first do not have number sense. Your textbook probably has lessons on Doubles and Make a 10, but if children do not have the four relationships Van de Walle (2006) discusses which build a child's number sense, then those problems in the book just become tasks to complete, not a strategy that will be used long term. Baroody (2009) provides evidence that a children's weak number sense often is due to their inadequate informal knowledge about numbers. We have children whose informal experience with numbers differ widely before they even come to school. This book is built on the premise that if we spend time in the early grades letting children explore and develop their informal knowledge, it will build a child's number sense and increase their flexibility when working with numbers, then fluency will actually occur faster. Plus, by developing a sense of the relationships between numbers, children will be able to transfer that number sense when working with multi-digit numbers as well.

Number Lines vs. Number Paths

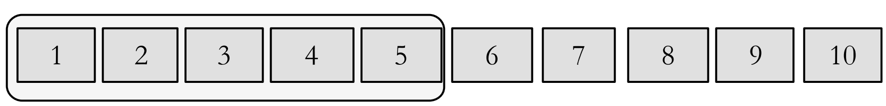
Many kindergarten and 1st grade classrooms have a number line along the classroom wall and on every child's desk. However, research has shown that *number lines* are conceptually too difficult for young children and instead we should be using number paths until a child is in 2nd grade (Fuson, et.al., 2009). A number path is a count model; the numbers are represented by a rectangle and each rectangle can be counted. A number line is a length model; each number is represented by its length from zero. With a number line, children have to count the length units and not the numbers. The number line model makes it difficult for young children to see the units, whereas in the number path, the units are easy to recognize. With a number line, some children start their count with zero and are off by one. Some count the space between the numbers, and when they end their count, their finger is between two numbers (count 4 spaces and your finger is between the 3 and 4) so they are unsure which number to use. Both of these errors do not occur when using a number path. Children in kindergarten and 1st grade are still making sense of numbers, we do not want to use models that confuse them; instead we want models that help solidify and build their understanding. For those reasons, in this book we use a number path instead of a number line as the iconic representation of what the child does in the enactive stage. Below are examples of a number path and a number line along with the understandings that children must have for the model to make sense (Fuson, et.al., 2009, p.44):

Count and Cardinal Word Meanings When Counting Things in a Number Path

Count word reference: "This rectangle is where I say five."

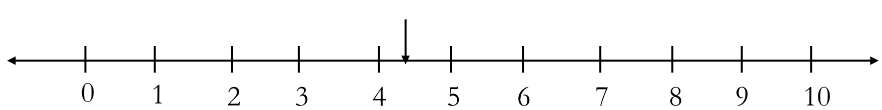


Cardinal word reference: "These are five rectangles."

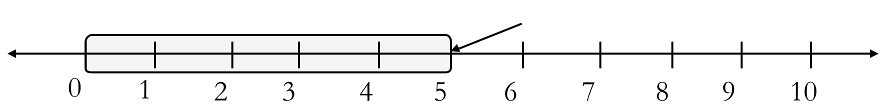


Count and Measure Word Meanings When Counting Unit Lengths on a Number Line

Count word reference: "This unit length is where I say five."



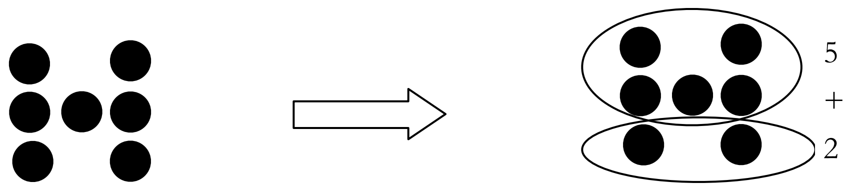
Measure word reference: "These are five unit lengths."



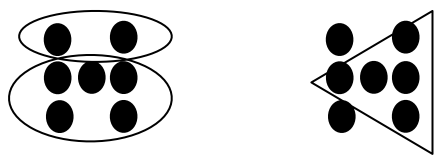
Spatial Relationships

The Power of The Relationship: Spatial Relationships

Close your eyes and think of “seven.” What did you see? The numeral ‘7’? The word *seven*? Or are you one of the few that has a picture of seven things? For many children the only picture they have of seven is the numeral. That “picture” makes it almost impossible for them to see how that can be broken into a 5 and a 2, or a 6 and a 1. However if a child sees this for seven, they can actually see a 5 and a 2:

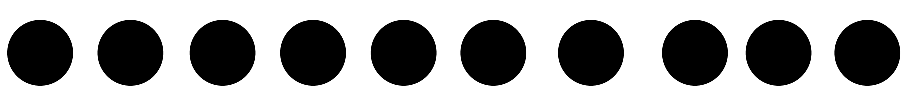


Using that same pattern, some children may see the 5 and 2 differently:

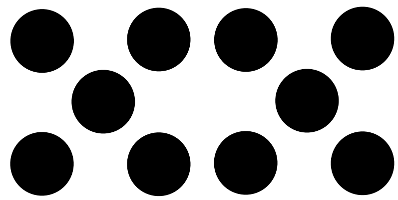


Even cooler is that they do not have to see it as only 5 and 2. Some may see 6 and 1 or 4 and 3 (Can you see them?). The point is to give them a visual representation of the abstract symbols we ask them to compute with. Many children are still grappling with making sense of single digits; that 7 can be 7+0, 6+1, 5+2, 3+4, and so on. However, at the same time they are dealing with this understanding, we are asking them to think about combining those same numbers but with place value: 1 and 6 now become 16. When we give them a visual picture of 1 + 6 they can see for themselves that it makes 7 and not just because we put a ‘+’ sign in between them.

Building spatial relationships also builds the idea of subitizing. Subitizing is being able to instantly recognize how many are in a set (Clements, 1999). For most people we can only subitize small sets (less than 5) unless they are in some kind of familiar pattern. For larger sets of numbers, we tend to decompose the set into smaller sets that we can instantly recognize. Look at the dots below and determine how many total dots there are:



Did you do it without having to count every single dot to make sure of your total? Now try this one:



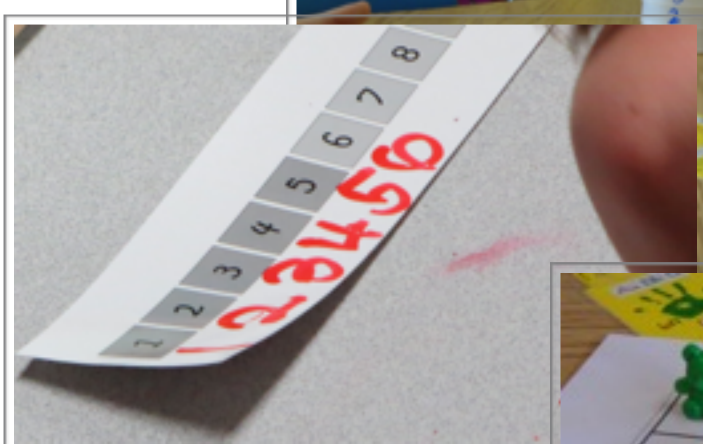
With this pattern you can instantly recognize the familiar pattern of five and there are two of them, so there has to be ten. Why are these dots so important to developing number sense and fluency? Think about the problem 7 + 6. Many adults think about that problem as 6 + 6 + 1, but to young children they do not see how a 7 can become 6 +1. To them it is just a bunch of symbols on paper, but what if they had a spatial relationship built for each of those numbers, maybe like the one below, do you think they could 'see' how 7 + 6 = 6 + 6 + 1?

Spatial relationships give children the ability to manipulate numbers in ways they may not yet be able to do with abstract numerals. Developing lots of visual pictures to correspond with the numeral helps children to move past seeing a number as a set of individual items and instead see groups within the set.

Along with dot patterns, children should also have practice with subitizing using their fingers. However, dot patterns and fingers are not the only way to build spatial relationships. A number's placement along a number path helps give children a visual picture that relates a number to other numbers. By showing the placement in a number path, children will start to understand the magnitude of a number; 9 is much larger than 2, but 5 and 6 are close to each other. A number path is a powerful tool for children which leads into a number line that children can use even as they progress into upper elementary and into their learning of fractions.

The ten-frame and the MathRacks™ are other tools that build spatial relationships, but they also help children with the benchmarks of 5 and 10. So they are discussed more in that section of this book.

One/Two More & Less



The Power of The Relationship: One/Two More and Less

Knowing one/two more or less allows children to be flexible thinkers and aides in mental computation. For instance, if a child understands that 9 is one less than 10, when they see 9+5 they can think to themselves "That is like 10+5, which is 15 so it is just one less." Not understanding this relationship limits children to seeing 9+5 as just 9+5, and if they do not remember that "fact", they revert back to counting on their fingers. This then leads into the same kind of calculations when dealing with multidigit addition where all too often we see children still counting on their fingers instead of using relationships. When children are presented with a problem like 59 + 25, we often see them adding 9 + 5 on their fingers then adding 5 + 2 + 1. If instead children have the relationship built of One/Two More and Less, it is faster, (and dare I say easier) than the traditional algorithm, to think of the problem as 60 + 25 and then take one away.

On the next page is the addition fact chart. If we teach the addition facts as isolated facts (learn +0, +1, +2, +3…), children have 121 facts to remember. If instead we focus on four types of facts and on building the relationship of One/Two More and Less we cover all 121 'facts.' The larger benefit of allowing children to learn their facts in this way is that it becomes so much easier for children to pull facts out of their memory. Try to memorize this eleven digit number: 25811141720. Do you think you have it? You might for right now, but will you be able to remember it in three days, how about two months? If instead I tell you to look for a relationship within the numbers (2 5 8 11 14 17 20), can you remember them now? What if I tell you that if you start with the first number and add 3 you get the next number and keep adding 3; can you remember all the numbers now? I have done this activity with a lot of teachers, and it is amazing to see that they remember this eleven digit number months after I first give it to them. When we focus on how things are connected it becomes easier to retrieve them from our memory. This is the way children learn facts; some can memorize and do just fine, others see relationships between facts, and others cannot memorize and do not see relationships, so we have to explicitly bring those relationships out for children. The four types of facts to focus on are highlighted in the chart, the dark color is the 'fact' and the lighter version of that color is +/- one or two:

- Purple – plus zero
- Orange – doubles
- Green – facts that make 10
- Blue – 10 plus something

+	0	1	2	3	4	5	6	7	8	9	10
0	0	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10	11
2	2	3	4	5	6	7	8	9	10	11	12
3	3	4	5	6	7	8	9	10	11	12	13
4	4	5	6	7	8	9	10	11	12	13	14
5	5	6	7	8	9	10	11	12	13	14	15
6	6	7	8	9	10	11	12	13	14	15	16
7	7	8	9	10	11	12	13	14	15	16	17
8	8	9	10	11	12	13	14	15	16	17	18
9	9	10	11	12	13	14	15	16	17	18	19
10	10	11	12	13	14	15	16	17	18	19	20

Another issue children sometimes struggle with is a big idea known as hierarchical inclusion. This is the idea that if I have 7 of something I also have 6, 5, 4, etc; 7 includes all the numbers below it. You can observe this when you ask a child questions like, "*We have 6 cupcakes and 4 kids. Do I have enough cupcakes so that each child can have one?"* Some children do not understand that 6 *includes* 4, but that idea will start to be developed as you work with activities to help children build the relationship of One/Two More and Less.

Benchmarks of 5 and 10

The Power of The Relationship: Benchmarks of 5 and 10

The numbers 5 and 10 are important benchmarks in our number system because we use a Base 10 and 2 fives make a ten. Helping children see how numbers relate to 5 and 10 becomes useful as they start to compute with numbers. If you know that 7 is 5 + 2 or it is three less than 10, think about how that could help children as they solve these problems: 7+8, 5 + 2, 7 – 2, 7 – 3, 13 – 7, 7 + 3, 57 + 3, 68 + 7, 47 + 6 (Van de Walle, 2006). How would you use the benchmarks of 5 and 10 to help you solve those problems? Here are two examples:

7 + 8

5 2

5 + (2+8)

13 – 7

3 4

(13-3) - 4

That was just one possible way to use 5 and 10 to solve these problems; there are other ways as well. Children tend to receive lots of experiences decomposing numbers, but what is lacking is knowing how that knowledge can help them and which way they should decompose a number. For example I may know all the ways to decompose the number 6 (5+1, 4+2, 3+3, etc), but when I am presented with the problem 6 + 9 do I know how decomposing the 6 can be helpful and which decomposition of 6 I should use to help me solve the problem? If I understand that I can break apart the 6, and I know 9's relationship to 10, I can instantly see that I should decompose the 6 into a 5 and 1 instead of a 4 and 2 (or 3 and 3).

Carpenter, et al (1999) highlight the stages young children go through when learning to solve addition and subtraction problems.

Direct Modeling → Counting → → → Derived Fact → Fact

It is a large leap for some children to get from the counting phase to the derived fact stage because making that leap means the child can break the problem into parts they know instead of counting one-by-one:

> A child who is in the counting phase solves 8 + 7 like this; they hold "8" in their head then to add 7 they count one-by-one, *"9, 10, 11, 12, 13, 14, 15."*

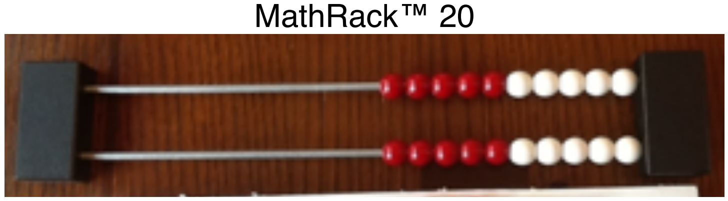
A child who can use derived facts may still start with the 8, but uses groups to add the 7; *"8 plus 2 is 10. I know 7 can be a 2 and 5. I've added 2, so I need to add 5 more which makes it 10 + 5…15."*

There is a lot of number sense that children need in order to build their derived facts. Many children still do not see groups within a number, to them 7 is just 7 (1,2,3,4,5,6,7) it is not a 5 and a 2, and they do not know that it is three away from 10. The activities in this section help children see how numbers relate to five and ten. The Spatial Relationships section helps children see all types of groupings for a number which is important, but the groupings of five and ten are extremely important. One of the most common derived facts we try to help children understand is the "make a ten" strategy. However, if they do not know a number's relationship to 10, they cannot use the "make a ten" strategy.

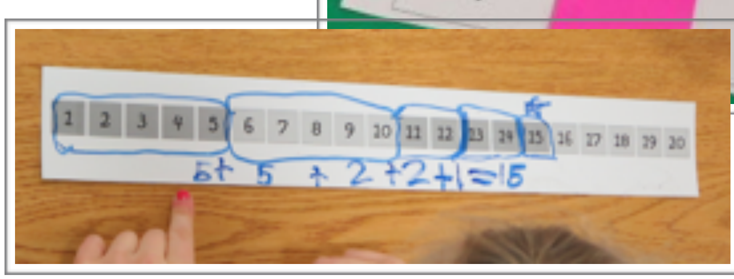
Even if a child can "make a 10" does not mean they will use it as an efficient strategy. For example, with the 8 + 7 problem, the child might have been able to break the 7 into a 2 and 5 to make 8 + 2 + 5. However, this is only helpful if they know 10 + 5. Children who do not know the "10 plus something" idea will count on one-by-one to add 5 to the 10. Time needs to be spent helping children understand the numbers 11 through 19 as "10 plus something." This idea is the beginning of helping children understand place value. Young children do not see the "1" as being any different just because we moved it over two centimeters; think of the number "11," why is one of the "1s" any different from the other??? Children need experiences with seeing objects grouped into ten and some more (example: 17 cubes separated into 10 cubes and 7 cubes) and relating that to the written form of the number.

Doing activities with ten frames and MathRacks™ are a great way to help build this relationship. If you do not have MathRacks™ (also known as rekenreks), get some! If you can not purchase MathRacks™, make your own.

MathRack™ 20



Part-Part-Whole



The Power of The Relationship: Part-Part-Whole

The three prior relationships help build the Part-Part-Whole concept; the idea that a number (7) is not just seven items. It can be decomposed into many different parts. Early on, children use this concept with the numbers up to 10; 7 can be 6 + 1, 5 + 2, etc. As their mathematical education continues, this idea is useful as they start to deal with multidigit numbers (37 can be 30+7, 32+5, 20+17, etc.) and fractions (7 is the same as 6 ½ + ½ or 5 ¾ + 1 ¼ and so on) and on into algebraic reasoning (x + 5 = 7).

A child can become accurate and efficient with basic facts through memorization. However, that knowledge can only be applied to that particular task (Baroody, et.al, 2009). A child who has only memorized 8 + 7 = 15 does not see the connection to the problem 38 + 7. When we focus on the relationships in this book, it helps give children flexibility when dealing with their basic facts and extending that knowledge to new tasks. Children who can decompose numbers, understand a number's relationship to 5 and 10, and know one/two more and less will see both 8 + 7 and 38 + 7 in the same way:

Child 1: *"8 + 2 makes 10 and add 5 more makes 15."*

"38 + 2 makes 40 and add 5 more makes 45."

Child 2: *"8 + 7 is like 7 + 7 which is 14, plus one more makes 15."*

"38 + 7 is like 37+ 7 which is 44, plus one more makes 45."

Child 3: *"8 + 7 is like (5 + 3)+(5 + 2). I put the two 5s together to make 10 and add the 3 and 2 gives me 15."*

"38 + 7 is like (35 + 3)+(5 + 2). I put the 35 and 5 together to make 40 and add the 3 and 2 to give me 45."

All of these strategies should be accepted and encouraged. When we build a child's number sense, it promotes *thinking* instead of just *computing*. Too many children (and even adults) compute without thinking, thus they do not notice when an answer is unreasonable. The most common error with multidigit addition is when a child does not 'carry the one,' they just add and write the numbers down:

$$\begin{array}{r} 29 \\ \underline{+17} \\ 316 \end{array}$$

If a child *thinks* before they *compute*, they might think to themselves *"29 is just one away from 30. If I add one to 29 (to make 30) I need to add 16* *more. So it is 30 + 16 which is 46."* In order to think in that way a child must understand a number's part-part-whole relationship; that a number can be decomposed into parts and will still stay the same.

This idea can be a large leap for some children. Basically we are asking children to think of 8 + 7 as the same as 8 + 2 + 5. For a child who believes 7 is just 7, they do not understand how that 7 became a 2 and 5 or that separating the 7 into a 2 and 5 did not change the fact that they still have seven total. Children may have no problem seeing that they need to add 2 to the 8 to make 10, but they may have a problem knowing the result of taking away 2 from the 7 leaves them with 5 (Baroody, et.al, 2009). The more practice children have with decomposing numbers into their parts, in meaningful and fun ways, the more automatic it becomes.

Appendix

Assessments

These assessments are meant to help you determine areas of need, not to monitor a child’s progress. All the assessment items in this book can be given to a small group at the same time, or you can assess one individual at a time.

Administering the Spatial Relationship Assessment:

The items that say "when shown a dot pattern card," you do not have to only use the dot pattern cards. Use the MathRack™ or ten frame cards to vary the spatial picture that the child sees. This is a way to make sure they can recognize an 8 (or any other number) whether it is a dot pattern, in the ten frame, or on a MathRack™.

Assessment Task	What instructor says	Possible child responses
1. When asked to represent 4 using their fingers, the child produces 4 fingers without having to count them.	*"Show me four using your fingers."*	Level 1: Child does not show 4 fingers Level 2: Child puts one finger up at a time saying *"1,2,3,4"* and shows 4 Level 3: Child puts all four fingers up at once without having to count individually
2. When shown the familiar dice dot patterns for 1 through 6, the child instantly tells how many without having to count each individual dot. *(do use dot pattern cards for this assessment item)*	Flip over a dot pattern that contains the dice pattern for any number 1-6 and ask the child, *"How many dots are on this card?"*	Level 1: Child does not give an accurate response Level 2: Child counts each dot one-by-one to get the correct answer Level 3: Child instantly recognizes the amount without counting and says the correct number within 3 seconds
3. When shown a number using a dot pattern card (you can use a ten frame or MathRack™ also) for 3 seconds, the child can reproduce the same number with counters, their fingers, dot stickers, or some other manipulative.	*"I am going to show you a picture, and I want you to use your fingers or these counters to create the same number of items you see in the picture."*	*If the child does not start working you can prompt them by saying "How many dots did you see?"* Level 1: Child cannot produce the same number Level 2: Child shows the same number, but does not position them the same. Level 3: Child puts their cubes in the same exact position as the dots.
4. When asked to represent 8 using their fingers, the child produces 8 fingers	*"Show me eight using your fingers."*	Level 1: Child does not show 8 fingers Level 2: Child puts one finger up at a time saying *"1,2,3,4..."* until he/she has 8 Level 3: Child puts up all five on one hand without counting, but then counts *"6,7,8"* on the other hand Level 4: Child puts all 8 fingers up at once without having to count any
5. When shown a dot pattern card (or ten frame/MathRack™) containing a number larger than 6, the child can determine the number without having to count each individual dot/item.	*"I am going to show you a picture, and I would like you to tell me how many dots/items you see."*	Level 1: Child does not give the correct number Level 2: Child counts each dot/item one-by-one to get the correct answer Level 3: Child recognizes one of the groups and counts on from that. Level 4: Child instantly recognizes the amount without counting and says the correct number within 3 seconds

Spatial Relationship Assessment Checklist: Date Given ___________
Write "yes" or "no" in the box for each child along with any relevant comments.

	Child 1	Child 2	Child 3	Child 4	Child 5
1. When asked to represent 4 using their fingers, the child produces 4 fingers without having to count them.					
2. When shown the familiar dice dot patterns for 1 through 6, the child instantly tells how many without having to count each individual dot. *(use dot pattern cards)*					
3. When shown a dot pattern card for 3 seconds, the child can reproduce the same number with counters, their fingers, dot stickers, or some other manipulative.					
4. When asked to represent 8 using their fingers, the child produces 8 fingers by either instantly showing 8 or by showing 5 then counting 6, 7, 8. *(indicate in the box which strategy they used to show 8)*					
5. When shown a dot pattern card larger than 6, the child can determine the number without having to count each individual dot/item.					

Administering the One/Two More/Less Assessment

For each item a specific number is given, but you can use **smaller numbers if the child does not get it right** the first time or you can **extend the assessment by doing the same activities with numbers in the teens.**

Assessment Task	What Instructor Says	Possible child responses
When shown a numeral, like 7, and asked "What number is one more than this number?" the child answers correctly. *(if they can do this use another number and ask what number is two more)*	Flip over a 3x5 card that has a numeral written on it (5-9) and say *"What number is one more than this number?"* If the child answers correctly flip over the next number and ask *"What number is two more than this number?"* Notate in the checklist if they can do +1 and +2	Level 1: Child will give an incorrect answer (try a smaller numeral) Level 2: Child can give correct answer for +1 Level 3: Child does +1 but has to count to figure out +2 Level 4: Child can give correct answer for +1 and +2 Level 5: Child can give correct answer when presented with a numeral in the teens
Ask the child to count out 5 cubes. After the child has successfully done that, add one cube to the pile and ask how many cubes there are now. Does the child know there are 6 without recounting them all? *(if the child can do this ask a similar question but add TWO)*	*"Please count out 5 cubes."* If the child cannot count 5 cubes, do it together so the child can see the 5 cubes being counted. Then add one cube to the pile and say, *"I added one cube to your pile, how many do you have now?"* If the child answers correctly say, *"Please count out 7 cubes… now I added two cubes to your pile, how many cubes are there?"*	Level 1: Child answers incorrectly Level 2: Child gives the correct number of cubes but has to recount all the cubes. Level 3: Child knows how many when you add one without having to recount. Level 4: Child knows how many when you add one, but recounts when you add two. Level 5: Child knows how many when one or two are added.
When shown a numeral, like 6, and asked "What number is one less than this number?" the child answers correctly. *(if the child can do this use another number and ask what number is two less)*	Flip over a 3x5 card that has a numeral written on it (5-9) and say *"What number is one less than this number?"* If the child answers correctly flip over the next number and ask *"What number is two less than this number?"* Notate in the checklist if they can do -1 and -2.	Level 1: Child will give an incorrect answer (try a smaller numeral) Level 2: Child can give correct answer for -1 Level 3: Child can give correct answer for -1 and -2 Level 4: Child can give correct answer when presented with a numeral in the teens
Ask the child to count out 9 cubes. After the child has successfully done that, take one cube away and ask how many cubes the child has now. Does the child know there are 8 without recounting them all? *(if the child can do this ask a similar question but take TWO away)*	*"Please count out 6 cubes."* If the child cannot count 6 cubes, do it together so the child can see the 6 cubes being counted. Then remove one cube from the pile and say, *"I took away one cube from your pile, how many do you have now?"* If the child can answer correctly say, *"Please count out 9 cubes…now I took away two cubes from your pile, how many cubes are there?"*	Level 1: Child answers incorrectly Level 2: Child gives the correct number of cubes but has to recount all the cubes. Level 3: Child knows how many when you remove one without having to recount. Level 4: Child knows how many when you remove one, but recounts when you remove two. Level 5: Child knows how many when one or two are removed.

One/Two More and Less Assessment Checklist: Date Given _______

Write "yes" or "no" in the box for each assessment item along with any relevant comments to help you determine the level (on the next page) of each child.

	Child 1	Child 2	Child 3	Child 4	Child 5
When shown a numeral, like 7, and asked "What number is one more than this number?" the child answers correctly. *(if they can do this use another number and ask what number is two more)*					
Ask the child to count out 5 cubes. After the child has successfully done that, add one cube to their pile and ask how many cubes the child has now. Does the child know there are 6 without recounting them all? *(if they can do this ask a similar question but add TWO)*					
When shown a numeral, like 6, and asked "What number is one less than this number?" the child answers correctly. *(if they can do this use another number and ask what number is two less)*					
Ask the child to count out 9 cubes. After the child has successfully done that, take one cube away and ask how many cubes the child has now. Does the child know there are 8 without recounting them all? *(if they can do this ask a similar question but take TWO away)*					

Administering the Benchmarks of 5 & 10 Assessment

Assessment Task	What Instructor Says	Possible child responses
Give the child a MathRack™. Ask the child to show you 7 beads on the MathRack™. Watch to see how the child counts out the 7.	*"I'm giving you a tool we will be using later in class. This tool has beads on it, please slide over 7 beads on one of the rows?"*	Level 1: cannot count 7 beads Level 2: counts the 7 beads one-by-one Level 3: grabs 5 beads and 2 more Level 4: counts three beads to leave and pushes over the rest
Count out 6 cubes and ask the child how many more cubes are needed to make 10.	Count out 6 cubes in front of the child, say to the child *"I have 6 cubes here, how many more do I need put out here so that I have 10 cubes?"* If child cannot do this task use a number path, put your finger on 6 and ask the child how many jumps you would have to do to get to the 10 spot.	Level 1: child cannot do the task Level 2: child adds cubes one-by-one to get to 10, then counts how many they added Level 3: child adds groups to get to ten (add 2, then 2) Level 4: child just knows you need to add 4
Count out 17 cubes and ask the child how many cubes you would need to take away to make 10.	Count out 17 cubes in front of the child, say to the child *"I have 17 cubes here, how many do I need to take away so that I have only 10 cubes?"* If child cannot do this task use a number path, put your finger on 16 and ask the child how many jumps you would have to go back to get to the 10 spot.	Level 1: child cannot do the task Level 2: child takes away cubes one-by-one to get to 10, then counts how many they took away or child might count on from 10. Level 3: child takes away groups to get to ten (take away 5, then 2) Level 4: child just knows you need to take away 7
Ask the child to count out 12 cubes and write the number on paper. Then ask the child to show you with the cubes what the '2' (point to it) and the '1' means with the cubes.	*"I have a bunch of cubes here; will you count out just 12 cubes for me, please?" "Great, now on this paper please write that number as big as you can." "So this number tells me how many cubes you have right here?"* (wait for confirmation from the child) Then point to the 2 and say *"Show me with your cubes what this number means. Put the cubes right here next to the number." "Okay, can you show me what the 1* (point to it) *means?"*	Level 1: child cannot show the two numbers with the cubes Level 2: child shows 2 cubes for the "2" and but 1 cube for the "1" Level 3: does level 2, but when asked about the remaining 9 cubes, the child figures out those go with the 1 to make a 10. Level 4: child shows the 2 cubes and the ten that represent the 1.

Benchmarks of 5 and 10 Assessment Checklist: Date Given ________

In each box write a brief note about what the child did for that assessment item for more details read “Administering the Assessment” on the next page.

	Child 1	Child 2	Child 3	Child 4	Child 5
Give the child a MathRack™. Ask the child to show you 7 beads on the MathRack™. Watch to see how the child counts out the 7.					
Count out 6 cubes and ask the child how many more cubes are needed to make 10.					
Count out 17 cubes and ask the child how many cubes you would need to take away to make 10.					
Ask the child to count out 12 cubes and write the number on paper. Then ask the child to show you with the cubes what the ‘2’ (point to it) and the ‘1’ means with their cubes.					

Administering the Part-Part-Whole Assessment

Assessment Task	What Instructor Says	Possible child responses
Count out 7 cubes, separate into two groups (like a 3 and 4). Does the child know there are still 7 cubes?	*"I am going to put out 7 cubes on the table."* Count them out loud so the child can hear you. *"How many cubes did I just count out?"* Wait for the child to tell you 7. *"I am going to put some of the cubes over here. How many cubes do I have altogether?"*	Level 1: child believes there is not 7 cubes Level 2: child has to count them to determine if there are still 7 Level 3: child knows that even though you moved some of them, there are still 7
Count out 8 cubes. Put some cubes in a cup so the child cannot see them. The child can see how many are still left, ask the child how many you put in the cup.	*"I am going to count out 8 cubes."* Count them out loud so the child can hear you. *"Now, I am going to put some of them in this cup to hide from you."* Grab 3 in your hand so the child cannot see how many you took and put them in the cup. *"Now we had 8, there are only these cubes left. How many cubes did I put in the cup?"*	Level 1: child cannot determine the amount you hid in the cup. (Try a smaller number, like 4, and see if the child can determine how many you hide.) Level 2: child counts one-by-one to figure out how many you hid. Level 3: child can tell you within 3 seconds how many you hid.
Using one of the dot pattern cards, cover some of the dots with a Post-it note so that they are hidden from the child. Tell the child how many dots are on the entire card, but some are covered by the Post-it. Can the child determine the number of hidden dots?	*"This card has 6 dots on it, but some of the dots are covered by this Post-it note. Can you tell me how many dots are covered by the Post-it?"*	Level 1: child does not give an accurate answer. (Try a smaller number, like 4, and see if the child can determine how many are covered.) Level 2: child counts one-by-one to figure out how many dots are hiding under the Post-it. Level 3: child can tell you within 3 seconds how many dots are hiding under the Post-it.

Part-Part-Whole Assessment Checklist: Date Given ____________

Write "yes" or "no" in the box for each assessment item along with any relevant comments.

	Child 1	Child 2	Child 3	Child 4	Child 5
Count out 7 cubes, separate into two groups (like a 3 and 4). Does the child know there are still 7 cubes?					
Count out 8 cubes. Put some cubes in a cup so the child cannot see them. The child can see how many are still left, ask the child how many you put in the cup.					
Using one of the dot pattern cards, cover some of the dots with a Post-it note so that they are hidden from the child. Tell the child how many dots are on the entire card, but some are covered by the Post-it. Can the child determine the number of hidden dots?					

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